The invariant form of the generators of semisimple Lie and quantum algebras in their arbitrary finite-dimensional representation

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Abstract

An explicit form of the generators of quantum and ordinary semisimple algebras for an arbitrary finite-dimensional representation is found. The generators corresponding to the simple roots are obtained in terms of a solution of a system of matrix equations. The result is presented in the form of $N_l \times N_l$ matrices, where N_l is the dimension of the corresponding representation, determined by the invariant Weyl formula.

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1 Introduction

In the present paper we pursue the main idea of the recent paper of the author [1] (apart from some inessential technical details) to use the character Weyl formula [2] for the construction in explicit form of the generators of both the quantum (and usual) semisimple algebras in an arbitrary finite-dimensional representation. In the construction of [1] the main idea was the proposition to use the Weyl character formula for the calculation of the result of the action of the group element $\exp\tau = \exp\sum_1^r \tau_i h_i$ (h_i are the Cartan elements of the algebra) on the basis state vectors of a finite-dimensional representation. This gives the possibility in the equations, defining a quantum algebra:

$$R_i X_j^{\pm} = \pm \tilde{K}_{j,i} X_j^{\pm} R_i, \quad [X_i^+, X_j^-] = \delta_{j,i} \frac{R_i - R_i^{-1}}{2\sinh w_i t}, \quad R_i \equiv \exp w_i t h_i \tag{1.1}$$

to consider "group elements" R_i in a given representation l as known and $X_i^{\pm}$ as the finite dimensional matrices with the known structure to be determined. In (1.1) K is the Cartan matrix, $K_{j,i} w_i = w_j K_{j,i} \equiv \tilde{K}_{j,i}$, t is the deformation parameter.

From the first $2r^2$ equations of (1.1) it is possible to obtain the selection rules, which define the structure of the $X_i^{\pm}$ matrices. The second r^2 equations define $X_i^{\pm}$ matrices uniquely up to an orthogonal transformation, the sense of which will be explained below.

The present paper is devoted to a realization of the program described above.

The paper is organized in the following way. In section 2 we describe the connection between the Weyl character formula and and the result of the action the group element $\exp\tau$ on the basis vectors of an arbitrary finite-dimensional representation l. In section 3 the selection rules and explicit form of the equations which the "primitive" matrix elements of generators $X_i^{\pm}$ are satisfy are presented. The meaning of the orthogonal transformation and its role in the construction is discussed in section 4. The consequences which follow as the result of combining the selection rules with orthogonal invariance are discussed in the section 5. The explicit solution for generators of each simple root in its canonical form is presented in the section 6. In section 7 the main result in the form of the factorization theorem is proved which restricts the solution of the whole problem to the case of the simple algebras $A_2, B_2 = C_2, G_2$. In section 8 the corresponding problem is solved for the algebras of the second rank. The general case of an arbitrary semisimple algebra is considered in section 9. Concluding remarks and prospects for further consideration are gathered in section 10.

2 Weyl character formula and action of the group element $\exp\tau$ on basis vectors of an arbitrary finite-dimensional representation

Throughout this paper we make no distinction between the bases of the quantum and the usual semisimple algebras keeping in mind that for their finite-dimensional representations, their dimensions and characters (in the usual case) are described by the famous Weyl formulae [2].

The basis state vectors of a finite-dimensional irreducible representation of a semisimple algebra $l = (l_1, l_2, ..l_r)$ are constructed by repeated application of the lowering X_i^- generators to the highest vector $\mid l\rangle$ with the properties:

$$X_s^+ \mid l\rangle = 0, \quad h_s \mid l\rangle = l_s \mid l\rangle$$

where $X_s^{\pm}, h_s$ are the generators corresponding to the simple roots and Cartan elements respectively.

The obvious difficulty of such a construction consists in the fact that not all state vectors arising in this way are linearly independent and an additional procedure for excluding linearly independent components with further orthogonalization of the basis is necessary. Usually this is not a simple problem.

Nevertheless, the values which the group element $\exp\tau \equiv \exp\sum h_i\tau_i$ takes on the basis vectors may be obtained from the invariant Weyl character formula for an irreducible representation $l = \sum h_i l_i$.

The Weyl character formula:

$$\pi^l(\exp\tau) = \frac{\sum_W \delta_W \exp(\tau_W, l + \frac{1}{2}\rho)}{\sum_W \delta_W \exp(\tau_W, \frac{1}{2}\rho)} \tag{2.2}$$

presented in form of the sum of exponents (the denominator is always a divisor of the numerator!):

$$\pi^l(\exp\tau) = \sum_{n^k}^{N_l} C_{n^k} \exp(\tau, n^k) = \sum_{n^k}^{N_l} C_{n^k} \exp\sum_i^r (\tau_i n_i^k)$$

gives the answer to many questions about the structure of the basis of the corresponding representation. In (2.2) W is the element of the discrete Weyl group, δ_W its signature, τ_W result of the action group element W on τ, C_{n^k} is the multiplicity of the corresponding exponent and finally, $\frac{1}{2}\rho$ is half the sum of the positive roots of the algebra.

We remind the reader of the definition:

$$\pi^l(\exp\tau) = \mathrm{Trace}(\exp\tau) = \sum_\alpha^{N_l} \langle\alpha \mid (\exp\tau) \mid \alpha\rangle,$$

where $\langle\alpha\mid, \mid\alpha\rangle$ are the bra and ket basis state vectors of the representation l.

Comparing the last two expressions we see that Weyl formula gives the answer to the question about the action of group element $\exp\tau$ on basis vectors with a given number of lowering operators of different kinds. Indeed:

$$\exp\tau(X_1^-)^{m_1}...(X_r^-)^{m_r} \mid l\rangle = \exp\sum\tau_i l_i \exp - \sum_{p,s} m_p \tilde{K}_{p,s}\tau_s (X_1^-)^{m_1}...(X_r^-)^{m_r} \mid l\rangle \tag{2.3}$$

(of course the order of the lowering operators is inessential in the last expression).

Equating each exponent of the Weyl formula to (2.3) we can without any difficulty find the indices m_i corresponding to it. The only thing, that Weyl formula cannot do is to distinguish between basis vectors with the same number of lowering operators of the same kind taken in different order. Nevertheless, it defines the number of such states (i.e. the multiplicity of the corresponding exponent in the formula for the character) These comments play a key role in the proposed construction.

3 Selection rules and the system of equations for "primitive" matrices

Let us introduce generators $\bar{h^i}$ dual to h_i with the properties:

$$[\bar{h^i}, X_j^{\pm}] = \pm\delta_{i,j} X_j^{\pm}, \quad \mathrm{Trace}(h_j, \bar{h^i}) = \delta_{i,j}$$

Generators $\bar{h}^i$ may be expressed as a linear combination of Cartan elements:

$$\bar{h} = hK^{-1}$$

This fact can be checked by a simple direct computation.

The simplest and shortest way to calculate the values, which the group element $\exp(\bar{h}, p) = \exp \sum \bar{h}^i p_i$ takes on the basis vectors of the finite-dimensional representation consists in the exchange in the formulae of the previous section $\tau \to K^{-1}p$.

Let us now consider the family of the mutual commutative group elements

$$\bar{R}_i = \exp \bar{h}^i t, \quad R_i = \prod_1^r (\bar{R}_s)^{\tilde{K}_{s,i}}.$$

The eigenvalues of each such element and their multiplicity can be calculated with the help of the Weyl character formula and have the form:

$$C_{n^k} \exp t(n^k K^{-1})_i.$$

Thus each basis vector of the finite-dimensional representation l may be marked with the help of the r-th dimensional vector $q^k = (q_1^k, q_2^k, ...q_r^k), q_i^k = e^{(n^k K^{-1})_i}$. The multiplicity of the corresponding vectors q^k we denote by $N_k (\equiv C_{n^k})$.

Now let us consider the first system of $2r^2$ equations (1.1). Rewritten in the terms of $\bar{R}^i$ generators, it takes the form:

$$\bar{R}^i X_j^{\pm} = \exp \pm \delta_{i,j} t X_j^{\pm} \bar{R}^i \tag{3.4}$$

and means that the matrix elements of the matrices $(X_i^{\pm})_{k,k'}$ are different from zero, when the indices k, k' are connected by the relation:

$$q_i^k = q_i^{k'} e^{\pm \delta_{i,j} t}, \quad \ln q_i^k - \ln q_i^{k'} = \pm (0, ...t, ..., 0) \equiv \pm j \tag{3.5}$$

where the single element t different from zero of the jth vector on the left- hand side sits in the j-th place. The r -th dimensional vector with components $\ln q_i^k$ we denote by the single symbol k.

The last r^2 equations (1.1) in the notation introduced above can be rewritten as:

$$(X_i^+)_{k,k-i} (X_j^-)_{k-i,k-i+j} - (X_j^-)_{k,k+j} (X_i^+)_{k+j-i} = \delta_{i,j} \frac{\sinh \ln R_i^k}{\sinh w_i t} I_{N_k} \tag{3.6}$$

where $(X_i^{\pm})_{k,k}$ are rectangular $N_k \times N_{k'}$ matrices which we will call "primitive" ones; I_{N_k} is $N_k \times N_k$ unit matrix.

We emphasize once more that values of all components of vectors q^k and their corresponding multiplicity are known from the Weyl character formula (2.2) and in the system (3.6) must be considered as a given. The unknowns are matrix elements of the rectangular primitive matrices of the given dimension.

4 Orthogonal symmetry

It follows from the Weyl character formula (2.2) that the multiplicity of each exponent in its involved will remain the same after reduction of the group element $\exp \tau$ to an arbitrary

subgroup of the initial semisimple group. This means, that all group elements $R_i, \bar{R}^i$ will have the same block structure – the multiplicity of each exponent in the corresponding place of each will be the same. (This doesn't exclude the possibility of the additional degeneracy, when the eigenvalues of some of elements R_i will be the same for different diagonal blocks.) Thus each element R_i is invariant with respect to similarity transformations generated by each $G(N_k, R)$ subgroups. The matrices $X_i^{\pm}$ preserve their form invariance, transforming in accordance with the law:

$$X_i^{\pm} \to G(N_k, R) X_i^{\pm} G^{-1}(N_k, R)$$

or on the level of primitive matrices:

$$(X_i^{\pm})_{k,k'} \to G(N_k, R)(X_i^{\pm})_{k,k'} G^{-1}(N_{k'}, R)$$

From the defining equations of the quantum algebra (1.1), their invariance with respect to an inner automorphism follows:

$$(X_i^-)^T \to X_i^+, \quad (X_i^+)^T \to X_i^-, \quad R_i^T = R_i$$

The same is true also with respect to the system, which arises after taking into account the selection rules which the primitive matrices satisfy.

In what follows we will find generators of the simple roots of quantum algebras satisfying the additional conditions:

$$(X_i^-)^T = X_i^+$$

In this case all equations involved preserve their invariance only after reduction of the direct product of linear groups on a direct product of orthogonal ones for which $O^{-1} = O^T$.

5 The selection rules in combination with the orthogonal symmetry

Let us consider separately the quantum algebra of the j-th simple root with corresponding generators $X_j^{\pm}, R_j$. The selection rules of section 3 for generators $X_j^{\pm}$:

$$\Delta \bar{h}^i = \pm \delta_{j,i}$$

have as their direct corollary the corresponding selection rules for the Cartan elements h_i:

$$\Delta h_i = \pm K_{j,i} \tag{5.7}$$

since the matrix elements $(X_j^{\pm})_{k,k'}$ are different from zero if the indices k, k' are connected by the relation:

$$k - k' = \pm j$$

or in the language of the basis state vectors this means that the state $\mid k' \rangle$ is distinguished from $\mid k \rangle$ by the action of exactly one X_j^- generator. Provisionally it is possible to write this fact in symbolic form:

$$\mid k' \rangle = [X_j^- \mid k \rangle]$$

The quadratic brackets in the last relation mean that the state vector $\mid k \rangle$ is constructed as some linear combinations of the lowering generators of the basis state vector $\mid k \rangle$ by adding the operator X_j^- .

From (5.7) it follows immediately that the generator $(X_j^{\pm})_{k,k'}$ is a direct sum of the generators of quantum A_1^q algebras with known dimension of each component of this direct sum. The deformation parameter for the quantum algebra connected with the j-th simple root is equal to $w_j t$. So if we are able to present the explicit expression for matrix elements $(X_j^{\pm})_{k,k'}$ in the canonical form for the A_1^q algebra then we can be sure that it is distinguished from the real one only by an additional orthogonal transformation. If we would be able to find all "mixing" angles (the orthogonal matrices of the corresponding dimension) of such an orthogonal transformation using equations from (3.6) which connects the generators of the nearest two or three simple roots (in the cases of the $D_n, E_{6,7,8}$ series), then this would give the final solution to the problem. In next few sections we consider a number of concrete examples and present the solution for the general case of an arbitrary representation.

6 Solution of A_1^q equations in "canonical" form

Let us choose from the sequence of the basis vectors, enumerated by the r-th dimensional vector k those $k(p_i^s)$ ones, which are the highest weight vectors with respect to the representation p_i^s of the quantum algebra A_1^q (i- is the number of simple roots, p_i^s is the index of its $(2p_i^s+1)$ dimensional representation, s - is the ordering number of it). This means, that,

$$(X_i^+)_{k(p_i^s)+i,k(p_i^s)} = 0$$

(in other words the state $k(p_i^s)+i$ is absent among the exponents of the Weyl character formula).

In the mean time let us set aside all additional indices k, i, s inessential for the problem of this section and consider the equations of quantum algebra, denoting its states by the usual $l, -l \leq m \leq l$ notation. We have:

$$X^+_{m,m-1}X^-_{m-1,m} - X^-_{m,m+1}X^+_{m+1,m} = \frac{\sinh 2mt}{\sinh t} I_{N_m} \tag{6.8}$$

By N_m we denote the dimension of the vector with a given m. For the state of the highest weight, $(X^+_{l+1,l}=0)$, the corresponding equation takes the form:

$$X^+_{l,l-1}X^-_{l-1,l} = \frac{\sinh 2lt}{\sinh t} I_{N_l} \tag{6.9}$$

The only difference with respect to the usual quantum algebra case consists in the matrix character of $X^+_{l,l-1}$; in the usual case this is a single c-number function, in the case under consideration this is an $N_l \times N_{l-1}$ rectangular matrix. $N_l \leq N_{l-1}$ because at least N_l basis states of N_{l-1} belong to the $(2l+1)$ multiplet. With the help of an orthogonal transformation O_{N_l} from the left and $O_{N_{l-1}}$ from the right the $N_l \times N_{l-1}$ rectangular matrix may be presented in the form of an $N_l \times N_l$ square matrix with elements different from zero on its main antidiagonal. From (6.9) all these elements are equal to $X^+_{l,l-1} = (\frac{\sinh 2lt}{\sinh t})^{\frac{1}{2}}$. Thus the remaining $N_{l-1}-N_l$ basis vectors belong to $(l-1)$ representations of the quantum algebra as the highest weight vectors of these representations. By the same reasoning, $X^+_{l-1,l-2}$ is an $N_{l-1} \times N_{l-2}$ rectangular matrix, which can be expressed as an antidiagonal $N_{l-1} \times N_{l-1}$ square matrix. The first of its $N_{l-1}-N_l$ matrix elements (counting from the left lower corner) coincide with the matrix elements of quantum algebra A_1^q; $X^+_{l-1,l-2} = (\frac{\sinh 2(l-1)t}{\sinh t})^{\frac{1}{2}}$ in its $(l-1)$ representation. The next

N_l ones coincide with the values $X^+_{l-1,l-2} = (\frac{\sinh 2(l-1)t \sinh 2t}{\sinh^2 t})^{\frac{1}{2}}$, the same elements but in the l representation. The remaining elements (if any) are the highest weight vectors of the $(l-2)$-th representations and so on.

After the mid point there arises only one difference $N_{m+1} \leq N_m$ and everything is repeated in the opposite direction. The matrix X^+ is symmetrical with respect to reflection in its main antidiagonal.

In the case when multiplicities of all representations states are the same $N_l = N_{l-1} = ... = N_{-l}$ the canonical form of the generators $X^{\pm}$ is a similarity transform of the unit matrix of the corresponding dimension multiplying matrix elements of the scalar quantum A^q_1 algebra. This case will be required in section 9 for proving the factorization theorem.

The matrix X^+ constructed by above rules we will call the canonical one and denote it by $\hat{X}^+$ with the all necessary indices which we have omitted in the beginning of this section.

7 Factorization of the problem

It follows from the results of the previous sections that the solution of the problem would be found if we would be able to find the orthogonal mixing angles connecting all neighbouring dots (roots) on the Dynkin diagram of the corresponding semisimple algebra. This problem in its turn can be achieved by the same calculations in the case of algebras only of the second rank. Indeed let i and $i+1$ be adjacent dots on the Dynkin diagram. Let us reduce the Weyl character formula on the subgroup of the second rank generated by the simple roots. The result is as follows: the representation l of the initial semisimple algebra is decomposed into the direct sum of irreducible representations of $G^2_{i,i+1}$ ($i, i+1$ are the simple roots generating this second rank algebra). If we know the orthogonal mixing angles for all (p,q) representations of such groups of the second rank we can resolve corresponding equations in the form [1]:

$$X^+_i = \hat{X}^+_i, \quad X^+_{i+1} = \prod_{p,q} O^{(p,q)}(i,i+1)\hat{X}^+_{i+1}$$

where the product is taken over orthogonal mixing angles of all representations (p,q) into which the initial representation l is decomposed under the reduction described above.

Let us begin from the first simple root on the Dynkin diagram. We have consequently

$$X^+_1 = \hat{X}^+_1, \quad X^+_2 = \prod_{p,q} O^{(p,q)}(1,2)\hat{X}^+_2$$

$$X^+_2 = \prod_{p,q} O^{(p,q)}(1,2)\hat{X}^+_2, \quad X^+_3 = \prod_{p,q} O^{(p,q)}(1,2)\prod_{p,q} O^{(p,q)}(2,3)\hat{X}^+_3$$

The next steps and the explicit form of the generators X^+_i are completely obvious.

Thus we reduce the solution of the problem with respect to an arbitrary semisimple algebra to the same problem with the respect to the algebras of only second rank. This assertion can be called the factorization theorem. We will return to this question in section 9.

[1] We write the element of orthogonal group only from the left omitting corresponding element O^{-1} from the right.

8 Algebras of the second rank

In this case the basis vectors are labeled by a pair of indices (the vector k is two dimensional). We denote its components for the highest weight vector by α, β.

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = k^{-1} \begin{pmatrix} p \\ q \end{pmatrix}$$

where (p, q) is the standard notation for an irreducible representation of the of the second rank algebras $A_2, B_2 = C_2, G_2$. The highest weight vector has unit multiplicity. All other basis vectors may be decomposed on subspaces with a given number N of lowering generators. Each basis vector of this subspace may be denoted by the natural index i $(1 \leq i \leq N)$ and "fine structure" index m taking all natural values between its multiplicity (determined by the Weyl character formula) and unity. Thus for such a vector we use the notation

$$(\alpha - i, \beta - N + i)(m)$$

The maximal available values for the multiplicity may be obtained from the following consideration. Let us for the meantime set aside the problem of linear independent components. Then at each next step after application of the two generators $X^-_{1,2}$ to each basis vector of the previous step the number of basis vectors arising will be twice that for the previous step and thus the maximal values of the multiplicity will be exactly the binomial coefficients $C^N_{\alpha-i,\beta-N+i}$.

Let us consider the first few lowest basis vectors in the meantime forgetting about their fine structure indices. We have the following chain:

$$(\alpha, \beta), [(\alpha - 1, \beta), (\alpha, \beta - 1)], [(\alpha - 2, \beta), (\alpha - 1, \beta - 1), (\alpha, \beta - 2)],$$

$$[(\alpha - 3, \beta), (\alpha - 2, \beta - 1), (\alpha - 1, \beta - 2), (\alpha, \beta - 3)], ...$$

where the generators of the subspaces with the given N $(0, 1, 2, 3, ...)$ are gathered in square bracket. The highest vector state $(\alpha, \beta) \equiv \mid p, q\rangle$ is annihilated by the both lowering generators and so is simultaneously the highest vectors of the p, q representations of the $A_1(1, 2)$ algebras generated respectively by the simple roots generators $X^{\pm}_{1,2}$. The state $X^-_2 \mid p, q\rangle$ with the respect to the $A_1(1)$ algebra is the highest vector of the representation $p - k_{21} \equiv p + k$ $(k = 1, 2, 3)$ and with the respect to $A_1(2)$ belongs to its q-th representation. The same situation takes place with respect to all other basis vectors and we will denote this fact by the additional upper, lower indices describing the multiplet structure of the corresponding basis states.

Thus the full notation which we use for the basis vectors states is the following:

$$(\alpha - i, \beta - N + i)^{p+N-i}_{q+i}(m)$$

In this notation the first few first basis vectors take the form

$$(\alpha, \beta)^p_q, [(\alpha - 1, \beta)^p_{q+1}, (\alpha, \beta - 1)^{p+k}_q], [(\alpha - 2, \beta)^p_{q+2}, (\alpha - 1, \beta - 1)^{p+k}_{q+1}, (\alpha, \beta - 2)^{p+2k}_q],$$

$$[(\alpha - 3, \beta^p_{q+3}, (\alpha - 2, \beta - 1)^{p+k}_{q+2}, (\alpha - 1, \beta - 2)^{p+2k}_{q+1}, (\alpha, \beta - 3)^{p+3k}_q], ...$$

After this preliminary comment we pass to concrete calculations. After the application of the equation $[X^+_1, X^-_2] = 0$ to the basis vectors of the first subspace $(N = 1)$ taking into account the selection rules of the section 4 is equivalent to a single equation:

$$(X^+_1)_{(\alpha,\beta-1),(\alpha-1,\beta-1)}(X^-_2)_{(\alpha-1,\beta-1),(\alpha-1,\beta)} = (X^-_2)_{(\alpha,\beta-1),(\alpha,\beta)}(X^+_1)_{(\alpha,\beta),(\alpha-1,\beta)}$$

Except for the vector $(\alpha-1,\beta-1)$ the maximal multiplicity of which is equal to 2, all other vectors involved are singlets and so taking into account the definition of of the canonical form of the quantum algebra generators of the section 6 and the definition of orthogonal matrices of section 5 we rewrite the last equation:

$$\sqrt{\frac{\sinh(p+k)t}{\sinh t}}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\cos\phi_1 & \sin\phi_1\\ -\sin\phi_1 & \cos\phi_1\end{pmatrix}\sqrt{\frac{\sinh(q+1)kt}{\sinh kt}}\begin{pmatrix}1\\0\end{pmatrix}=\sqrt{\frac{\sinh pt\sinh qkt}{\sinh t\sinh kt}}$$

or

$$\cos\phi_1=\sqrt{\frac{\sinh pt\sinh qkt}{\sinh(p+k)t\sinh(q+1)kt}},\quad \sin\phi_1=\pm\sqrt{\frac{\sinh((p+qk+k)t\sinh kt}{\sinh(p+k)t\sinh(q+1)kt}}$$

Thus we have calculated the two (dimensional) orthogonal mixing angle and multiplicity 2 for the state vector $(\alpha-1,\beta-1)$. Only in the case (we assume $q\leq p$ but this absolutely inessential) $q=0$ this multiplicity is degenerated up to unity as follows from the explicit expression for the mixing angle above. We emphasize that in the last calculations we have not used Weyl formula but have calculated the multiplicity of the basis state $(\alpha-1,\beta-1)$ independently.

For further calculations it will be more suitable to change notation for the basis vectors states and express them only in terms of the basis vectors of two $A_{(1,2)}$ algebras. It is not difficult check that the basis vectors of the N-th subspace may be enumerator with the help only one index $0\leq s\leq N$ as follows:

$$((p+ks)_{N-s},(q+N-s)_s)$$

In this notation the few first basis vector states have the form:

$$(p_0,q_0),[(p_1,(q+1)_0),((p+k)_0,q_1)],[(p_2,(q+2)_0),((p+k)_1,(q+1)_1),((p+2k)_0,q_2)],$$

$$[(p_3,(q+3)_0),((p+k)_2,(q+2)_1),((p+2k)_1,(q+1)_2),((p+3k)_0,q_3)]..$$

$$[(p_4,(q+4)_0),((p+k)_3,(q+3)_1),((p+2k)_2,(q+2)_2),((p+3k)_1,(q+1)_3),((p+4k)_0,q_4)]..$$

where l_k is symbolically equivalent to $l_k\equiv(X^-)^k\mid l\rangle$ (l is the index of representation of the A_1 algebra, k its quantum number).

In this language the conditions of the the mutual commutativity of the generators X_1^+ and X_2^- take the "factorizable" form (generators $X_{1,2}^{\pm}$ have matrix elements only between the states designated by indices p and q respectively):

$$(X_1^+)_{(p+ks)_{N-s},(p+ks)_{N-s+1}}(X_2^-)_{(q+N-s+1)_s,(q+N-s+1)_{s-1}}=$$
$$(X_2^-)_{(q+N-s)_s,(q+N-s)_{s-1}}(X_1^+)_{(p+k(s-1))_{N-s},(p+k(s-1))_{N-s+1}} \tag{8.10}$$

The last equations applied to three basis vectors of the $N=2$ subspace lead to the pair of equations:

$$(X_1^+)_{(p+k)_1,(p+k)_2}(X_2^-)_{(q+2)_1,(q+2)_0}=(X_2^-)_{(q+1)_1,(q+1)_0}(X_1^+)_{p_1,p_2}$$
$$(X_1^+)_{(p+2k)_0,(p+2k)_1}(X_2^-)_{(q+1)_2,(q+1)_1}=(X_2^-)_{q_2,q_1}(X_1^+)_{(p+k)_0,(p+k)_1}$$

It is necessary to solve the last equations under the additional conditions that the multiplicity of the states $(p+k)_1,(q+1)_1$ is equal to 2 and the maximal multiplicity of the states $(p+k)_2,(q+2)_1,(p+2k)_1,(q+1)_2$ are is equal to 3. Using the notation $n((q+2)_1),\tilde{n}^2((q+1)_2)$

for three- dimensional orthogonal matrices we obtain for their matrix elements the following values:

$$n^1_1=-\sqrt{\frac{\sinh(p-1)t\sinh(p+qk+k)t\sinh 2t\sinh kt}{\sinh(p+k)t\sinh(p+k-2)t\sinh(q+2)kt\sinh t}}$$

$$n^1_2=\sqrt{\frac{\sinh(p-1)t\sinh qkt\sinh pt}{\sinh(p+k)t\sinh(p+k-1)t\sinh(q+2)kt}}$$

$$\tilde{n}^1_1=\sqrt{\frac{\sinh((p+qk+k)t\sinh 2kt}{\sinh(p+2k)t\sinh(q+1)kt}},\quad \tilde{n}^2_1=\sqrt{\frac{\sinh pt\sinh(q-1)kt}{\sinh(p+2k)t\sinh(q+1)kt}}$$

From the last expression we see that in the first case we really have "saturation" – the multiplicity coincides with it maximal possible value, except for the case of the A_2 algebra ($k=1$). In this case and in all cases concerning the $\tilde{n}$ matrix we have:

$$(n^1_1)^2+(n^1_2)^2=1,\quad (\tilde{n}^1_1)^2+(\tilde{n}^2_1)^2=1$$

and we can assume that the multiplicity in these cases is equal to 2. The equation for the real values of the multiplicities in these cases will be clarified after explicit calculations of all other matrix elements of the $n,\tilde{n}$ matrices. (We don't consider here the cases of the possible additional degeneracy connected with particular values of the parameters (p,q) defining the representation).

In the case $N=3$ the main equations (8.10) take the form

$$\begin{aligned}(X^+_1)_{(p+k)_2,(p+k)_3}(X^-_2)_{(q+3)_1,(q+3)_0}&=(X^-_2)_{(q+2)_1,(q+2)_0}(X^+_1)_{p_2,p_3}\\ (X^+_1)_{(p+2k)_1,(p+2k)_2}(X^-_2)_{(q+2)_2,(q+2)_1}&=(X^-_2)_{(q+1)_2,(q+1)_1}(X^+_1)_{(p+k)_1,(p+k)_2}\\ (X^+_1)_{(p+3k)_0,(p+3k)_1}(X^-_2)_{(q+1)_3,(q+1)_2}&=(X^-_2)_{q_3,q_2}(X^+_1)_{(p+2k)_0,(p+2k)_1}\end{aligned}\tag{8.11}$$

The multiplicities of the states $(p+2k)_1,(q+1)_2$ are equal to 2, $(p+k)_2(q+2)_1$ – to 3, $(p+k)_3,(q+3)_1,(p+3k)_1,(q+1)_3$ – not more than 4, $(p+2k)_2,(q+2)_2$ – not more than 6.

In terms of canonical forms of the generators $X^{\pm}_{1,2}$ from the section 6 and orthogonal matrices of the sections 4,5 we rewrite the first equation (8.11):

$$\begin{pmatrix}0&0&\sqrt{\frac{\sinh(p+k-3)t\sinh 2t}{\sinh^2 t}}&0\\ 0&\sqrt{\frac{\sinh(p+k-2)t\sinh 3t}{\sinh^2 t}}&0&0\\ \sqrt{\frac{\sinh(p+k-4)t}{\sinh t}}&0&0&0\end{pmatrix}O_4\sqrt{\frac{\sinh(q+3)kt}{\sinh kt}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=$$

$$n\sqrt{\frac{\sinh(q+2)kt}{\sinh kt}}\begin{pmatrix}1\\0\\0\end{pmatrix}\sqrt{\frac{\sinh(p-2)t\sinh 3t}{\sinh^2 t}},$$

where O_4 is a 4×4 orthogonal matrix, n a 3×3 orthogonal matrix, the components of which were partly calculated at the previous step.

The last equation allows to calculate only the matrix elements of the first column of the O_4 matrix:

$$O^1_1=\sqrt{\frac{\sinh(q+2)kt\sinh(p-2)t\sinh 3t}{\sinh(q+3)kt\sinh(p+k-4)t\sinh t}}n^1_3$$

$$O^1_2=\sqrt{\frac{\sinh(q+2)kt\sinh(p-2)t}{\sinh(q+3)kt\sinh(p+k-2)t}}n^1_2$$

$$O^1_3=\sqrt{\frac{\sinh(q+2)kt\sinh(p-2)t\sinh 3t}{\sinh(q+3)kt\sinh(p+k-3)t\sinh 2t}}n^1_1$$

It is not difficult to check by direct computation that in the case of A_2 algebra ($k=1$) the dimension of O_4 matrix is reduced to O_2, in the case of $B_2=C_2$ (k=2) it reduced to O_3.

The second equation from (8.11) with the same comments as in the previous case takes the form,

$$\begin{pmatrix} 0 & \sqrt{\frac{\sinh(p+2k-1)t\sinh 2t}{\sinh^2 t}} & 0... \\ \sqrt{\frac{\sinh(p+2k-2)t}{\sinh t}} & 0 & 0... \end{pmatrix} P \begin{pmatrix} 0 & 0 & \sqrt{\frac{\sinh qkt}{\sinh kt}} \\ 0 & \sqrt{\frac{\sinh qkt}{\sinh kt}} & 0 \\ \sqrt{\frac{\sinh(q+1)kt\sinh 2kt}{\sinh^2 kt}} & 0 & 0 \end{pmatrix} n^{-1}=$$

$$\tilde n \begin{pmatrix} 0 & \sqrt{\frac{\sinh(q-1)kt}{\sinh kt}} \\ \sqrt{\frac{\sinh qkt\sinh 2kt}{\sinh^2 kt}} & 0 \end{pmatrix}\phi_1^{-1}\begin{pmatrix} 0 & \sqrt{\frac{\sinh(p+k-1)t\sinh 2t}{\sinh^2 t}} & 0 \\ \sqrt{\frac{\sinh(p+k-2)t}{\sinh t}} & 0 & 0 \end{pmatrix}$$

In the last equations it is known that the dimension of the orthogonal matrix P is less than 6. But it is possible to determine only its 6 elements (express them in terms of matrix elements of 3×3 orthogonal matrix n. We present result of neither cumbersome nor, on the other hand, very short calculations in the form:

$$\begin{pmatrix} \sqrt{\frac{\sinh(q+1)kt\sinh 2kt}{\sinh^2 kt}}P^3_2 & \sqrt{\frac{\sinh qkt}{\sinh kt}}P^2_2 & \sqrt{\frac{\sinh qkt\sinh 2t}{\sinh kt}}P^1_2 \\ \sqrt{\frac{\sinh(q+1)kt\sinh 2kt}{\sinh^2 kt}}P^3_1 & \sqrt{\frac{\sinh qkt}{\sinh kt}}P^2_1 & \sqrt{\frac{\sinh qkt\sinh 2t}{\sinh kt}}P^1_1 \end{pmatrix}=$$

$$\begin{pmatrix} \sqrt{\frac{\sinh(q-1)kt\sinh 2kt\sinh(p+k)t\sinh(p+k-1)t}{\sinh^2 kt\sinh(p+2k)t\sinh(p+2k-1)t}} & 0 \\ \sqrt{\frac{\sinh pt\sinh(p+kq+k)t\sinh(p+k-1)t\sinh 2t}{\sinh t\sinh(p+k)t\sinh(p+2k)t\sinh(p+2k-2)t}} & -\sqrt{\frac{\sinh qkt\sinh(p+2k)t(p+k-2)t}{\sinh kt\sinh(p+k)t\sinh(p+2k-2)t}} \end{pmatrix}\begin{pmatrix} n^1_2 & n^2_2 & n^3_2 \\ n^1_1 & n^2_1 & n^3_1 \end{pmatrix}$$

In particular,

$$P^3_2=\sqrt{\frac{\sinh(q-1)kt\sinh(p+k)t\sinh(p+k-1)t}{\sinh(q+1)kt\sinh(p+2k)t\sinh(p+2k-1)t}}$$

$$P^3_1=\frac{\sinh 2kt}{\sinh t\sinh kt}\sqrt{\frac{\sinh qkt\sinh 2t\sinh(p+qk+k)t\sinh(p-1)t}{\sinh(q+2)kt\sinh(p+2k)t}}$$

The third equation from (8.11) takes the form (we have multiplied it addition ally from the right on 2×2 orthogonal matrix $\tilde n$):

$$\sqrt{\frac{\sinh(p+3k)t}{\sinh t}}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}R_4\begin{pmatrix} 0 & \sqrt{\frac{\sinh(q-4)kt\sinh 3kt}{\sinh^2 kt}} \\ \sqrt{\frac{\sinh(q-2)kt\sinh 2kt}{\sinh^2 kt}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}=$$

$$\sqrt{\frac{\sinh(q-1)kt\sinh 3kt}{\sinh^2 kt}}\sqrt{\frac{\sinh(p+2k)t}{\sinh t}}\begin{pmatrix}1 & 0\end{pmatrix}\tilde n$$

In the last equation we have assumed the saturation case of the matrix R. But explicit calculation leads a the different result:

$$R_1^2 = \sqrt{\frac{\sinh(p+qk+k)t \sinh 3kt}{\sinh(p+3k)t \sinh(q+1)kt}}, \quad R_1^1 = \sqrt{\frac{\sinh(q-2)kt \sinh pt}{\sinh(p+3k)t \sinh(q+1)kt}}$$

from which we can conclude that the multiplicity is equal to 2.

From the results of the present section we conclude that asymptotically (the values of (p,q) are sufficiently large) the multiplicities of the spaces with the different N are the following:

$$N=1 \to (1,1), \quad N=2 \to (1,2,1), \quad N=3 \to (1,3,2,1), \quad N=4 \to (1,4,4,2,1)$$

and this result is obtained without using the Weyl character formula. When the parameters of the representation reach some particular values the multiplicity changes by a jump, which is it possible to observe from the structure of the orthogonal matrix connected with the corresponding basis vector.

8.1 The case of $(2,1)$ representation of B_2 algebra

From this concrete example the reader will be able to understand more clearly the main steps of the calculations and convinced himself of the self consistency of the whole construction.

The dimension of the representation under consideration is equal to 35. 35 exponents of the Weyl character formula (the explicit expression for the character of the (p,q) representation of the B_2 algebra can be found in Appendix I) may be distributed into subspaces with upper index N $(0 \leq N \leq 10)$ with the corresponding multiplicities in the following order:

$$[e^{(2\tau_1+\tau_2)}], [e^{\tau_2}, e^{(4\tau_1-\tau_2)}], [e^{(-2\tau_1+3\tau_2)}, 2e^{2\tau_1}], [3e^{\tau_2}, e^{(4\tau_1-2\tau_2)}], [2e^{(-2\tau_1+2\tau_2)}, 3e^{(2\tau_1-\tau_2)}],$$

$$[e^{(-4\tau_1+3\tau_2)}, 3, e^{(4\tau_1-3\tau_2)}],$$

$$[3e^{(-2\tau_1+\tau_2)}, e^{(2\tau_1-2\tau_2)}], [e^{(-4\tau_1+2\tau_2)}, 3e^{-\tau_2}], [2e^{-2\tau_1}, e^{(2\tau_1-3\tau_2)}], [e^{(4\tau_1-\tau_2)}, e^{-2\tau_2)}], [e^{(-2\tau_1-\tau_2)}]$$

The same basis vectors in terms of irreducible representations of the A_1 algebra, distributed into the same subspaces appear as follows:

$$[p_0, q_0], \quad [(p_1, (q+1)_0), ((p+k)_0, q_1)], \quad [(p_2, (q+2)_0), \phi_1((p+k)_1, (q+1)_1],$$

$$[l((p+k)_3, (q+2)_1), ((p+2k)_1, (q+1)_2)], \quad [\phi_2(p+k)_3, (q+3)_1), m((p+2k)_2, (q+2)_2)],$$

$$[((p+k)_4, (q+4)_1), n((p+2k)_3, (q+3)_2), ((p+3k)_2, (q+2)_3)],$$

$$[o((p+2k)_4, (q+4)_2), \phi_3((p+3k)_3, (q+3)_3)], \quad [((p+2k)_5, (q+5)_2), r((p+3k)_4, (q+4)_3)],$$

$$[\phi_4((p+3k)_5, (q+5)_3), ((p+4k)_4, (q+4)_4)], [((p+3k)_6, (q+6)_3), ((p+4k)_5, (q+5)_4)], [((p+4k)_6, (q+6)_4)]$$

The notations ϕ_s, l,m,n,o,r mean 2-th, 3-th dimensional orthogonal matrices respectively, which is necessary to find.

Of course in all expressions above it is necessary to put $p=2, q=1, k=2$. We preserve the previous notation to give to the reader the possibility of a simpler comparison with the results of the previous section. The following comment is necessary. Not all multiplets above begin from the zero index. So if a multiplet begins for instance from the first index $(p+k)_1$, this means that this is the multiplet with $l=2$ but not with $l=4$ and so on.

The condition of the mutual commutativity of $X_1^{\pm}$ generators with $X_2^{\mp}$, after equating to zero all matrix elements in the basis constructed above (it follows from the selection rules that the products of generators $X_1^+X_2^-$ and $X_2^-X_1^+$ conserve N and so it is necessary to check the matrix elements of the commutator with the fixed values of N) leads to the a system of equations to be solved. Below we present this system in abstract form, its encoding with the help of the formulae of Appendix II and explicit solutions are obtained at each step of the calculation. (Of course all results connected with $N=1,2,3$ may be obtained with the help the direct substitution $k=2,p=2,q=1$ into the general formulae obtained in the previous section, but for the fullness of the picture we present them once more).

$$N=1$$

$$(X_1^+)_{(p+k)_0,(p+k)_1}\phi_2(X_2^-)_{(q+1)_1,(q+1)_0}=(X_2^-)_{q_1,q_0}(X_1^+)_{p_0,p_1}$$

$$\sqrt{\frac{\sinh 4t}{\sinh t}}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\cos\phi_1 & \sin\phi_1\\ -\sin\phi_1 & \cos\phi_1\end{pmatrix}\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix}1\\0\end{pmatrix}=\sqrt{\frac{\sinh 2t}{\sinh t}}$$

or

$$\cos\phi_1=\frac{\sinh 2t}{\sinh 4t},\quad \sin\phi_1=\pm\frac{\sqrt{\sinh 6t\sinh 2t}}{\sinh 4t}$$

$$N=2$$

$$(X_1^+)_{(p+k)_1,(p+k)_2}(X_2^-)_{(q+2)_1,(q+2)_0}=(X_2^-)_{(q+1)_1,(q+1)_0}(X_1^+)_{p_1,p_2}$$

$$\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix}0 & \sqrt{\frac{\sinh 3t}{\sinh t}} & 0\\ 1 & 0 & 0\end{pmatrix} l\sqrt{\frac{\sinh 6t}{\sinh 2t}}\begin{pmatrix}1\\0\\0\end{pmatrix}=\phi_1\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix}1\\0\end{pmatrix}\sqrt{\frac{\sinh 2t}{\sinh t}}$$

$$l_1^1=\mp\sqrt{\frac{\sinh 2t}{\sinh 4t}},\quad l_2^1=\sqrt{\frac{\sinh t\sinh^2 2t}{\sinh 3t\sinh 4t\sinh 6t}},\quad l_3^1=\sqrt{\frac{\sinh 8t\sinh t}{\sinh 3t\sinh 6t}}$$

$$N=3$$

$$(X_1^+)_{(p+2k)_1,(p+2k)_2}(X_2^-)_{(q+2)_2,(q+2)_1}=(X_2^-)_{(q+1)_2,(q+1)_1}(X_1^+)_{(p+k)_1,(p+k)_2}$$

$$\sqrt{\frac{\sinh 4t}{\sinh t}}\begin{pmatrix}1 & 0 & 0\end{pmatrix} m\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ \frac{\sinh 4t}{\sinh 2t} & 0 & 0\end{pmatrix} l^{-1}=\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix}1 & 0\end{pmatrix}\phi_1^{-1}\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix}0 & \sqrt{\frac{\sinh 3t}{\sinh t}} & 0\\ 1 & 0 & 0\end{pmatrix},$$

$$m_1^3=\frac{\sinh 2t}{\sqrt{\sinh 4t\sinh 6t}}$$

$$m_1^2=(\cos\phi_1\sqrt{\frac{\sinh 3t}{\sinh t}}l_2^2-\sin\phi_1 l_1^2),\quad m_1^1=(\cos\phi_1\sqrt{\frac{\sinh 3t}{\sinh t}}l_2^3-\sin\phi_1 l_1^3)$$

$$N=4$$

$$(X_1^+)_{(p+2k)_2,(p+2k)_3}(X_2^-)_{(q+3)_2,(q+3)_1}=(X_2^-)_{(q+2)_2,(q+2)_1}(X_1^+)_{(p+k)_2,(p+k)_3}$$

$$\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix}0 & 0 & \sqrt{\frac{\sinh 3t}{\sinh t}}\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} n\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\phi_2^{-1}=$$

$$m\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ \frac{\sinh 4t}{\sinh 2t} & 0 & 0\end{pmatrix} l^{-1}\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix}0 & 1\\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0\\ 0 & 0\end{pmatrix}$$

In what follows the following notation will be convenient:

$$(N^1 \quad N^2) \equiv n \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \phi_2^{-1}$$

where $N^{1,2}$ are two three dimensional column vectors.

$$N = 5$$

$$(X_1^+)_{(p+2k)_3,(p+2k)_4} (X_2^-)_{(q+4)_2,(q+4)_1} = (X_2^-)_{(q+3)_2,(q+3)_1} (X_1^+)_{(p+k)_3,(p+k)_4},$$

$$(X_1^+)_{(p+3k)_2,(p+3k)_3} (X_2^-)_{(q+3)_3,(q+3)_2} = (X_2^-)_{(q+2)_3,(q+2)_2} (X_1^+)_{(p+2k)_2,(p+2k)_3}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \sqrt{\frac{\sinh 4t}{\sinh 2t}} n \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \phi_2^{-1} \sqrt{\frac{\sinh 4t}{\sinh t}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sqrt{\frac{\sinh 4t}{\sinh t}} (1 \quad 0) \phi_3 \sqrt{\frac{\sinh 4t}{\sinh 2t}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} n^{-1} = (0 \quad 0 \quad 1) m^{-1} \sqrt{\frac{\sinh 2t}{\sinh t}} \begin{pmatrix} 0 & 0 & \sqrt{\frac{\sinh 3t}{\sinh t}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The resolution of the last equations is the following:

$$\frac{\sinh 4t}{\sqrt{\sinh 2t \sinh 6t}} (N_1^1 \quad N_2^1 \quad N_3^1) = \left(o_3^1 \quad o_2^1 \quad \sqrt{\frac{\sinh 3t}{\sinh t}} o_1^1 \right),$$

$$\frac{\sinh 4t}{\sqrt{\sinh 2t \sinh 6t}} (R_1^2 \quad R_2^2 \quad R_3^2) = \left(m_3^3 \quad m_2^3 \quad \sqrt{\frac{\sinh 3t}{\sinh t}} m_1^3 \right)$$

where two three dimensional mutually orthogonal vectors R^1, R^2 are defined by the relation

$$(R^1 \quad R^2) = n \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \phi_3^{-1}$$

From the definition of $N^{1,2}, R^{1,2}$ vectors the following equation connects them:

$$(R^2 \quad R^1) = (N^1 \quad N^2) (\phi_2 + \phi_3)^{-1}$$

$$N = 6$$

$$(X_1^+)_{(p+3k)_3,(p+3k)_4} (X_2^-)_{(q+4)_3,(q+4)_2} = (X_2^-)_{(q+3)_3,(q+3)_2} (X_1^+)_{(p+2k)_3,(p+2k)_4}$$

$$\sqrt{\frac{\sinh 2t}{\sinh t}} \begin{pmatrix} 0 & \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 \\ 1 & 0 & 0 \end{pmatrix} r \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{\sinh 4t}{\sinh 2t} & 0 & 0 \end{pmatrix} o^{-1} =$$

$$\phi_3 \sqrt{\frac{\sinh 4t}{\sinh 2t}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} n^{-1} \sqrt{\frac{\sinh 2t}{\sinh t}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 & 0 \end{pmatrix} o \sqrt{\frac{\sinh 6t}{\sinh 2t}}$$

$$N = 7$$

$$(X_1^+)_{(p+3k)_4,(p+3k)_5} (X_2^-)_{(q+5)_3,(q+5)_2} = (X_2^-)_{(q+4)_3,(q+4)_2} (X_1^+)_{(p+2k)_4,(p+2k)_5}$$

$$\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix} 0 & 1 \\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 \\ 0 & 0 \end{pmatrix}\phi_4\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = r\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{\sinh 4t}{\sinh 2t} & 0 & 0 \end{pmatrix} o^{-1}\sqrt{\frac{\sinh 4t}{\sin t}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$N = 8$$

$$(X_1^+)_{(p+4k)_4,(p+4k)_5}(X_2^-)_{(q+5)_4,(q+5)_3} = (X_2^-)_{(q+4)_4,(q+4)_3}(X_1^+)_{(p+3k)_4,(p+3k)_5}$$

$$\sqrt{\frac{\sinh 2t}{\sinh t}}\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix} 1 & 0 \end{pmatrix}\phi_4^{-1} = \sqrt{\frac{\sinh 6t}{\sinh 2t}}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} r^{-1}\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix} 0 & 1 \\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 \\ 0 & 0 \end{pmatrix}$$

$$r_1^3 = \mp\sqrt{\frac{\sinh 2t}{\sinh 4t}}, \quad r_2^3 = \sqrt{\frac{\sinh t \sinh^2 2t}{\sinh 3t \sinh 4t \sinh 6t}}, \quad r_3^3 = \sqrt{\frac{\sinh 8t \sinh t}{\sinh 3t \sinh 6t}}$$

$$N = 9$$

$$(X_1^+)_{(p+4k)_5,(p+4k)_6}(X_2^-)_{(q+6)_4,(q+6)_3} = (X_2^-)_{(q+5)_4,(q+5)_3}(X_1^+)_{(p+3k)_5,(p+3k)_6}$$

$$\sqrt{\frac{\sinh 2t}{\sinh t}} = \sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\phi_4 & \sin\phi_4 \\ -\sin\phi_4 & \cos\phi_4 \end{pmatrix}\sqrt{\frac{\sinh 4t}{\sinh t}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or

$$\cos\phi_4 = \frac{\sinh 2t}{\sinh 4t}, \quad \sin\phi_4 = \pm\frac{\sqrt{\sinh 6t \sinh 2t}}{\sinh 4t}$$

It is more convenient to consider the system above grouping its equations into the pairs $(1,9),(2,8),(3,7),(4,6),(5,5)$.

From the first pair we find the two dimensional mixing angles $\phi_1 = \phi_4$. Equations $(2,8)$ allow us to determine the first and the last columns of the three dimensional orthogonal matrices l and r respectively and reconstruct their explicit form up to arbitrary two dimensional rotations [2] :

$$l = \phi_{12}^l \phi_{13}^l u_{2,3} \equiv \bar{l} u_{2,3}, \quad rW = \phi_{12}^l \phi_{13}^l u_{2,3} \equiv \bar{r} W v_{2,3},$$

where W is a three dimensional matrix with elements different from zero; ones on its main antidiagonal and g_{ij} are two dimensional rotation matrices in the (i,j) plane, parametrised by the parameter g;

$$\sin\phi_{1,2}^l = -\sqrt{\frac{\sinh 2t \sinh t}{\sinh 4t \sinh 5t}} \quad \sin\phi_{1,3}^l = -\sqrt{\frac{\sinh 8t \sinh t}{\sinh 3t \sinh 6t}},$$

$u_{2,3}, v_{2,3}$ two dimensional rotations in the $(2,3)$ plane with arbitrary parameters u and v.

Equations $(N = 3, N = 7)$ give the possibility to find the first rows of the matrices m, o and reconstruct them up to arbitrary two dimensional rotations:

$$m = w_{2,3}\phi_{12}^m \phi_{13}^m u_{1,2} \equiv w_{2,3}\bar{m} u_{1,2}, \quad oW = z_{2,3}\phi_{12}^m \phi_{13}^m v_{1,2} \equiv \bar{o} W v_{1,2},$$

where

$$\sin\phi_{1,2}^m = -\sqrt{\frac{\sinh 6t \sinh t}{\sinh 4t \sinh 5t}} \quad \tan\phi_{1,3}^m = -\frac{\sinh 2t}{\sinh 4t}\sqrt{\frac{\sinh 2t \sinh 5t}{\sinh t \sinh 8t}}$$

and the parameters (u,v) are the same as in the parametrisation of l, r matrices and(w,z) are arbitrary new ones.

[2] Equation $N = 8$ after its transposition for the unknown function rW coincides with the same for the l matrix. The same comments are true with respect to all pairs enumerated above.

Equations ($N=4, N=6$) determine the pair of mutual orthogonal vectors (N^1,N^2),(R^1,R^2) $((N^1)^2=(N^2)^2=(R^1)^2=(R^2)^2=1, (N^1N^2)=(R^1R^2)=0)$. The orthonormality conditions are not additional equations but a direct corollary of the previous equations for the matrices l,r,m,o .

$$(N^1,N^2)=\sqrt{\frac{\sinh 2t}{\sinh 4t}}w_{1,2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 & 0 \end{pmatrix}\bar{m}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{\sinh 4t}{\sinh 2t} & 0 & 0 \end{pmatrix}\bar{l}^{-1}\begin{pmatrix} 0 & 1 \\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 \\ 0 & 0 \end{pmatrix}\equiv w_{1,2}(\bar{N}^1,\bar{N}^2)$$

$$(R^1,R^2)=\sqrt{\frac{\sinh 2t}{\sinh 4t}}z_{1,2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 & 0 \end{pmatrix}\bar{o}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{\sinh 4t}{\sinh 2t} & 0 & 0 \end{pmatrix}\bar{r}^{-1}\begin{pmatrix} 0 & 1 \\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 \\ 0 & 0 \end{pmatrix}\equiv z_{1,2}(\bar{R}^1,\bar{R}^2)$$

We pay attention of the reader that the equations defined the (N,R) vectors contain only two (arbitrary up to now) parameters (w,z).

The equations $N=5$ connect N^1 (R^2) vectors with the first (third) columns of the matrices (o,m) respectively. We restrict ourselves to consideration of the first equation. The consequences of the second one are absolutely the same.

The vector N^1 and the first column of the matrix o are fully defined (as it follows from the explicit formulae for them above) up to orthogonal transformations $w_{1,2}, z_{2,3}$ respectively (rotation $v_{2,3}$, arising after using the equality $v_{1,2}W=Wv_{2,3}$, gives no output into first column of o matrix). With the help of the direct computation it is not difficult to convince that indeed:

$$\frac{\sinh 4t}{\sqrt{\sinh 2t\sinh 6t}}N^1_3=\sqrt{\frac{\sinh 4t\sinh 3t}{\sinh t\sinh 6t}}(\bar{m}^1_1\bar{l}^3_2+\bar{m}^2_1\bar{l}^2_2+\frac{\sinh 4t}{\sinh 2t}\bar{m}^3_1\bar{l}^1_2)=\sqrt{\frac{\sinh 3t}{\sinh t}}o^1_1$$

(all explicit values inside the relation to be proved are known) and

$$\frac{\sinh^2 4t}{\sinh 2t\sinh 6t}=(o^1_3)^2+(o^1_2)^2+\frac{\sinh 3t}{\sinh t}(o^1_1)^2$$

Thus among the three equations connected two three dimensional vectors N^1 and o^1 only one equation is essential. From this equation we obtain:

$$\cos(w+z)=\frac{\bar{o}^1_3\bar{N}^1_1+\bar{o}^1_2\bar{N}^1_2}{(\bar{N}^1_1)^2+(\bar{N}^1_2)^2}$$

The knowledge of the vectors N,R give

$$\cos(\phi_2+\phi_3)=\frac{\sqrt{\sinh 6t\sinh 2t}}{\sinh 4t}$$

So we have determined all parameters which permit the reconstruction in explicit form of the infinitesimal generators of the simple roots of the quantum B_2 algebra in its $(2,1)$ representation.

It is not surprising that in some of the cases we were able to obtain only some particular combination of parameters. Namely in this combination they arise in infinitesimal operators, which in the usual for quantum mechanical terminology are the observables in the problem under consideration. The reader can convinced from the corresponding formulae of the text of the truth of this assertion..

9 The general case of arbitrary semisimple (quantum) algebra

Let us consider the state vector of a finite dimensional representation (l) of an arbitrary quantum algebra with a fixed number of the lowering generators:

$$(X_1^-)^{m_1}(X_2^-)^{m_2}....(X_r^-)^{m_r} \mid (l_1,...l_r)\rangle$$

The order of the operators involved in the last formula is inessential. After the action (m_s+1) times on such a state vector by the generator X_s^+ we surely will obtain zero. But zero can arise at some intermediate step and answer as to when this question can be found after comparison with the Weyl character formula (2.2) with (2.3) as it was explained in section 1. Thus we can say that with respect to the representation of the algebra $A_1(s)$ the state vector above belongs to its $(l_s-\sum m_t k_{ts})$-th finite dimensional irreducible representation with the quantum number m_s. We would like emphasize once more that it may happen that the first j basis vectors of this representation is forbidden by Weyl character formula. In this case the index of the representation is reduced to jw_s, where w_s is the norm of the s - th simple root of the algebra under consideration.

Thus instead of a notation for basis vectors of the l representation with the help of r-dimensional vectors k (see section 5), it is possible and more convenient to use a notation connected with the indices and quantum numbers of the irreducible representations of r formally independent $A_1(s)$ algebras.

From now on we use the notation for basis vectors in the subspaces with fixed N (the number of the lowering generators):

$$[...(l_s-\sum_{t\neq s} m_t K_{ts})_{m_s})...], \quad \sum m_s = N$$

In connection with the equations defining a quantum algebra (1.1), the commutators of each pair $(i\neq j)$ of generators of positive (i), negative (j) simple roots are equal to zero. This condition written in the form of matrix elements of the corresponding commutator takes the form:

$$(X_i^+)_{(l_i-\sum_{t\neq i} m_t K_{ti}-K_{ji})_{m_i-1}),(l_i-\sum_{t\neq i} m_t K_{ti}-K_{ji})_{m_i}}(X_j^-)_{(l_j-\sum_{t\neq j} m_t K_{tj})_{m_j+1},(l_j-\sum_{t\neq j} m_t K_{tj})_{m_j}} =$$

$$(X_j^-)_{(l_j-\sum_{t\neq j} m_t K_{tj}+K_{ij})_{m_j+1},(l_j-\sum_{t\neq j} m_t K_{tj}+K_{ij})_{m_j}}(X_i^+)_{(l_i-\sum_{t\neq i} m_t K_{ti})_{m_i-1}),(l_i-\sum_{t\neq i} m_t K_{ti})_{m_i}} = 0 \qquad (9.12)$$

In fact this is exactly the equation (8.10) (rewritten in another notation with $k=-K_{2,1}$,$1=-K_{1,2}$) used in all concrete calculations of the section 8.

Let us consider first the equation (9.12) in the case $K_{i,j}=K_{j,i}=0$, the case when two different simple roots are connected by zero matrix elements of the Cartan matrix. We use the abbreviations $I=l_i-\sum_{t\neq i} m_t K_{ti}$, $J=l_j-\sum_{t\neq j} m_t K_{tj}$ and rewrite (9.12):

$$(X_i^+)_{I_{m_i-1},I_{m_i}}(X_j^-)_{J_{m_j+1},J_{m_j}} = (X_j^-)_{J_{m_j+1},J_{m_j}}(X_i^+)_{I_{m_i-1},I_{m_i}}$$

In these last relations the multiplicities of states (I_{m_i},J_{m_j}) are equal (these are only different notations for the same state vector); for the same reason multiplicities of each of the following pairs are equal (I_{m_i-1},J_{m_j+1}),(I_{m_i},J_{m_j+1}),(J_{m_j},I_{m_i-1}). This means that the matrix elements

of X_i^+, X_j^- are canonically equivalent under unitary transformation on matrix elements of the usual one-dimensional quantum algebra. See in this context the comments at the end of section 6.

Thus we conclude that the commutator of each pair of generators of positive-negative simple roots connected with zero valued elements of Cartan matrix are zero as a direct corollary of the selection rules of section 4.

So it is necessary to solve equations of commutativity only for the pairs of simple root generators connected by the elements of the Cartan matrix different from zero . In other words this means that at each step of calculations it is necessary to find the orthogonal mixing angles between each pair marked by dots on the Dynkin diagram.

The technique of these calculations is described in the section 8. The final result is presented in the recurrence formulae of the section 7.

The considerations of this section may be considered as proof of the main assertion: **The problem of the explicit form of the generators of simple roots of semisimple Lie and quantum algebras may be reduced to the solution of the same problem for the second rank algebras** .

10 Outlook

The line of approach suggested in the author's recent paper [1] has been transformed in the present one into a closed mathematical scheme, which allows us to obtain the explicit form of the generators of semisimple Lie and quantum algebras in their arbitrary finite-dimensional representations.

It turns out that for this purpose it is necessary to introduce new objects (to the best of our knowledge unknown up to now) - orthogonal matrices, dimensions of which coincide with the multiplicities of corresponding exponents in the Weyl character formula or, and this is equivalent, a number of linear independent basis vectors of representation, which can be constructed with a fixed number of the lowering operators of the same kind.

The calculation scheme of section 8 (see also examples of $(2,1),(1,1)$ representations of A_1 and B_2 algebras respectively, considered in [1]) allow us to obtain at every step the explicit expressions for certain "primitive" matrices (constructed with the help of orthogonal ones), from which generators of the corresponding simple roots are constructed. Subsequent application of this procedure to all dots of the Dynkin diagram solves the problem of finding in explicit form all generators of simple roots of the algebra under consideration in its arbitrary finite-dimensional representation (see section 7 and 9).

The most surprising and intriguing result is the universal character of the orthogonal matrices in the asymptotic region of sufficient large representation parameters. They do not depend on the basis of the representation dictated by Weyl character formula and can be calculated by independent means as was done in section 8 for subspaces of basis vectors with $N = 1, 2, 3, 4$.

The "limiting values" calculated in this way of orthogonal matrices have a universal analytical dependence (from matrix elements of Cartan matrix) for all algebras of the second rank, repeating by their degeneration properties the structure of corresponding discrete Weyl groups.

Observing the degeneracy properties of orthogonal matrices

the dimension changes discretely when passing from one representation to another one) it is possible to arrive at definite conclusions about the structure of the representation bases equivalent to the corollaries which follow from the Weyl character formula.

If it would it be possible to find independent arguments (based on representation or structure theory of semisimple algebras) explaining the fact of the existence and main properties of the orthogonal matrices, then the problem of the explicit form of generators would be solved with the help of corresponding formulae of sections 7 and 9. In this case it may be turn out that all representation theory of semisimple algebras is only a direct consequence of the corresponding theory of orthogonal matrices.

We would like to finish this outlook by two little comments of a different kind.

Firstly, there is no doubt that the construction of the present paper is applicable without essential alteration to the case of semisimple Lie and quantum super-algebras. Its solution is connected only with correct manipulations and computation in the Grassman space.

Secondly, some reminders about the history of the problem.

The way proposed in the present paper (see also [1]) is diametrically opposite to the usual used for solution of this problem beginning from the famous I.M.Gel'fand and M.L.Tsetlin papers [3]. The main idea of the previous investigations consisted in finding a family of mutually commutative operators (constructed from the generators of the algebra) the eigenvalues of which are able to define the basis of the representation. As the reader have seen no mention of such a family of operators (including the Casimir ones) was used in the proposed construction. Only facts from the global representation theory of semisimple algebras encoded in the Weyl character formula lead to introduction of orthogonal matrices with the corresponding algorithm for their calculations. In this the way of the present paper differs from previous investigations.

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11 Appendix I

The character of (p,q) representation of B_2 gives by the formula:

$$\pi^{p,q}(\tau_1,\tau_2)=\frac{\sinh l_1\tilde{\tau}_1 \sinh l_2\tilde{\tau}_2-\sinh l_1\tilde{\tau}_2 \sinh l_2\tilde{\tau}_1}{\sinh 2\tilde{\tau}_1 \sinh \tilde{\tau}_2-\sinh \tilde{\tau}_2 \sinh \tilde{\tau}_1}$$

where $l_1-l_2=p+1, l_2=l_2, l_1=p+q+2, l_1+l_2=p+q+3, \tilde{\tau}_1=\tau_1, \tilde{\tau}_2=\tau_2-\tau_1$. In the case under consideration $p=2, q=1$ the last expression pass to:

$$\pi^{2,1}(\tau_1,\tau_2)=16\cosh\tilde{\tau}_1\cosh\tilde{\tau}_2(\cosh?2\tilde{\tau}_1+\cosh\tilde{\tau}_1\cosh\tilde{\tau}_2+\cosh?2\tilde{\tau}_2)-12\cosh\tilde{\tau}_1\cosh\tilde{\tau}_2-1$$

The multiplication by terms of the last expressions leads to the sum of exponents of Weyl formula presented in the corresponding place of the main text.

12 Appendix II

Below we present the explicit canonical form of the generators $X^{\pm}_{1,2}$ in the case of $(2,1)$ representation of B_2 algebra. In all formulae $p=2, q=1, k=2$.

$$(X^+_1)_{p_0,p_1}=\sqrt{\frac{\sinh 2t}{\sinh t}}, \quad (X^+_1)_{p_1,p_2}=\sqrt{\frac{\sinh 2t}{\sinh t}}$$

$$(X_1^+)_{(p+k)_0,(p+k)_1}=\sqrt{\frac{\sinh 4t}{\sinh t}}\begin{pmatrix}1 & 0\end{pmatrix},\quad (X_1^+)_{(p+k)_1,(p+k)_2}=\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix}0 & \sqrt{\frac{\sinh 3t}{\sinh t}} & 0\\ 1 & 0 & 0\end{pmatrix},$$

$$(X_1^+)_{(p+k)_2,(p+k)_3}=\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix}0 & 1\\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0\\ 0 & 0\end{pmatrix},\quad (X_1^+)_{(p+k)_3,(p+k)_4}=\sqrt{\frac{\sinh 4t}{\sinh t}}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$(X_1^+)_{(p+2k)_1,(p+2k)_2}=\sqrt{\frac{\sinh 4t}{\sinh t}}\begin{pmatrix}1 & 0 & 0\end{pmatrix},\quad (X_1^+)_{(p+2k)_2,(p+2k)_3}=\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix}0 & 0 & \sqrt{\frac{\sinh 3t}{\sinh t}}\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix},$$

$$(X_1^+)_{(p+2k)_3,(p+2k)_4}=\sqrt{\frac{\sinh 2t}{\sinh t}}\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ \sqrt{\frac{\sinh 3t}{\sinh t}} & 0 & 0\end{pmatrix},\quad (X_1^+)_{(p+2k)_4,(p+2k)_5}=\sqrt{\frac{\sinh 4t}{\sinh t}}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

$$(X_1^+)_{(p+3k)_2,(p+3k)_3}=(X_1^+)_{(p+k)_0,(p+k)_1},\quad (X_1^+)_{(p+3k)_3,(p+3k)_4}=(X_1^+)_{(p+k)_1,(p+k)_2},$$

$$(X_1^+)_{(p+3k)_4,(p+3k)_5}=(X_1^+)_{(p+k)_2,(p+k)_3},\quad (X_1^+)_{(p+3k)_5,(p+3k)_6}=0(X_1^+)_{(p+k)_3,(p+k)_4},$$

$$(X_1^+)_{(p+4k)_4,(p+4k)_5}=\sqrt{\frac{\sinh 2t}{\sinh t}},\quad (X_1^+)_{(p+4k)_5,(p+4k)_6}=\sqrt{\frac{\sinh 2t}{\sinh t}}$$

$$(X_1^+)_{q_1,q_0}=1$$

$$(X_2^-)_{(q+1)_1,(q+1)_0}=\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix}1\\ 0\end{pmatrix},\quad (X_2^-)_{(q+1)_2,(q+1)_1}=\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix}1 & 0\end{pmatrix},$$

$$(X_2^-)_{(q+2)_1,(q+2)_0}=\sqrt{\frac{\sinh 6t}{\sinh 2t}}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix},\quad (X_2^-)_{(q+2)_2,(q+2)_1}=\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ \frac{\sinh 4t}{\sinh 2t} & 0 & 0\end{pmatrix}$$

$$(X_2^-)_{(q+2)_3,(q+2)_2}=\sqrt{\frac{\sinh 6t}{\sinh 2t}}\begin{pmatrix}0 & 0 & 1\end{pmatrix}$$

$$(X_2^-)_{(q+3)_2,(q+3)_1}=\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix},\quad (X_2^-)_{(q+3)_3,(q+3)_2}=\sqrt{\frac{\sinh 4t}{\sinh 2t}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

$$(X_2^-)_{(q+4)_2,(q+4)_1}=(X_2^-)_{(q+2)_1,(q+2)_0},\quad (X_2^-)_{(q+4)_3,(q+4)_2}=(X_2^-)_{(q+2)_2,(q+2)_1},$$

$$(X_2^-)_{(q+4)_4,(q+4)_3}=(X_2^-)_{(q+2)_3,(q+2)_2}$$

$$(X_2^-)_{(q+5)_3,(q+5)_2}=(X_2^-)_{(q+1)_1,(q+1)_0},\quad (X_2^-)_{(q+5)_4,(q+5)_3}=(X_2^-)_{(q+1)_2,(q+1)_1},$$

$$(X_2^-)_{(q+6)_4,(q+6)_3}=1$$

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